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The bond-diluted $Z(q)$ ferromagnetic model: criticality and break-collapse method

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Abstract. Within a real-space renormalization group (RG) which preserves two-site correlation functions, we study, on the square lattice, the criticality of the bond-diluted $Z(q)$ ferromagnetic model. We generalize the 'break-collapse method' which simplifies greatly the exact calculation of arbitrary $Z(q)$ two-terminal clusters (commonly appearing in RG approaches) mainly for a large value of q. We reproduce, in the pure case, several known exact results. The structure of the phase diagrams, for all the values of q, is obtained with a good precision. The massless spin–wave-like phase, which evolves into the Kosterlitz–Touless phase for $q \to \infty$, occurs around $q_c = 5$ (in agreement with the well known exact result). The structure of the phase diagrams in the diluted case is qualitatively similar to that obtained from the pure model. The massless spin–wave-like phase resists in an interval of concentration p which increases for the large values of q.

1. Introduction

The $Z(q)$ model, unifies in a single framework, a large amount of theoretically important statistical models. It generalizes the universal class of the clock model and as $q \to \infty$ it coincides with the planar XY model. It has attracted the attention of many researchers [1–16] mainly addressing the square lattice whose study is simplified because of the self-duality in the pure case which makes it possible to obtain some exact results at criticality [1, 3, 5, 10]. The $Z(q)$ model includes several models like the q-state Potts model, the cubic model, the Ashkin–Teller (A–T) model, etc. Several attempts have been made to construct the phase diagram of this model and to locate the value q_c of q above which the spin–wave-like phase [9] would appear. It has been argued by different techniques that this massless spin–wave phase starts around $q_c = 5$ [5, 12].

The diluted systems exhibit a very wide range of phenomena and provide a rich field of investigation. More results have been obtained on the diluted Ising spin systems which have been studied intensively [17].

It was shown exactly that the asymptotic behaviour in the percolation regime, $p \geqslant p_c$ is [18]

$$\frac{KT_c(p)}{2J} \propto \frac{1}{\ln[1/p - p_c)]} \tag{1}$$

where $T_c(p)$ is the critical temperature versus the bond concentration and P_c is the percolation concentration. This result implies that the thermal crossover exponent $\Phi_c = 1$. Consequently, the dilute Ising model belongs to the same universality class as the pure one.

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However, the q-state Potts model which generalizes the Ising model presents a crossover to a new diluted fixed point, and its critical expondent α is positive ($2 - \alpha = d\nu$) for $2 < q \leqslant 4$ [19]. The transition remains second order but exhibits new values of the critical exponents. This behaviour is in accord with the Harris criterion [20]. The bond-diluted $Z(4)$ model was investigated using a real-space renormalization group (RG) scheme [21]. The phase diagram and the crossover critical exponents are obtained. Since the pure model describes two coupled Ising models with two-spin coupling J_2 and four-spin coupling J_4, the quenched bond dilution does not affect the percolation concentration p_c of the system. The diluted A–T model, with two independent random variables, J_2 and J_4, distributed according to two different probabilities with respective concentration p_1 and p_2, was studied [22] using an extension of the Migdal–Kadanoff renormalization group (MKRG) method. It was shown that the model presents a percolation phase diagram which describes two types of links with different concentrations. For $T \neq 0$, a rich variety of phase diagrams was obtained.

The main purpose of this paper is to investigate the effects of a quenched bond dilution on $Z(q)$ model for $q > 5$ on the square lattice. We do this within a real-space renormalization-group framework [23, 24] based on the self-dual Wheatstone bridge cluster. This procedure has been shown to be very convenient for the renormalization group in the square lattice for the pure [23] and diluted q-state Potts models [19], discrete n-vector model [25] as well as for the pure isotropic [26] and anisotropic [27] $Z(4)$ models. However, we generalize the break-collapse method (BCM) [24] to the pure and bond-diluted planar discrete $Z(q)$ models to simplify the performance of tracing over all the possible configurations of the system.

Within this method, the percolation concentration is $p_c = \frac{1}{2}$ for all the values of the coupling ratios $K_\delta/K_1(\delta = 2, \ldots, [q/2])$ (where $[q/2]$ denotes the integer part of $q/2$), confirming the conjecture proposed by Alcaraz and Tsallis [7]. Using this technique we reproduce all the critical frontiers of the usual phases with very good precision and reproduce some exact results. In contrast to the MKRG method this one shows that the massless spin–wave-like phase occurs, in the pure case ($p = 1$), for $q \geqslant 5$, in agreement with the well known results. Since this method does not always reproduce the correct universality class, like most RG approaches, we will not indicate the order of transition at some fixed points.

In section 2 we introduce the model, develop the RG formalism and generalize the BCM to the $Z(q)$ model. Section 3 contains our main results, and finally, in section 4 we conclude.

2. The model and RG formalism

Consider a simple square whose lattice points are occupied by a classical 'spin' of unit length pointing in one of the equiangular coplanar directions which are given by the angles $(2\pi k/q)(k = 0, 1, \ldots, q-1)$. The interaction between the spins is assumed to depend only on the absolute value of the angular difference between the two nearest-neighbour (NN) spins. Its most general form can be written as [6]

$$-\beta H = \sum_{\langle ij \rangle} \sum_{m=1}^{[q/2]} K_m^{ij} \cos\left(\frac{2\pi}{q} m(\sigma_i - \sigma_j)\right). \quad (2)$$

Here, $\sigma_i = 0, 1, \ldots, q-1$ and $\langle ij \rangle$ denotes a pair of NN spins. The model defined in (2) is called the general discrete planar model or the $Z(q)$ model.

For special values of K_m^{ij}, the $Z(q)$ model is reduced to the q-state Potts model [28] defined as

$$-\beta H = \sum_{\langle ij \rangle} K_{ij}(q\delta_{\sigma_i\sigma_j} - 1)\,. \tag{3}$$

If the K_m^{ij} are all zero for $m > 1$, the model is the clock or vector Potts model while for $q = 4$, it is reduced to the Ashkin–Teller model [29].

A convenient variable, for renormalization-group study, is the transmissivity vector $t = (1, t_1, t_2, \ldots, t_{q-1})$, defined through

$$t_\delta^{ij} = t_{q-\delta}^{ij} = \frac{1 + 2\sum_{m=1}^{[q/2]}{}' \omega_m^{ij} \cos\left(\frac{2\pi}{q} m\delta\right)}{1 + 2\sum_{m=1}^{[q/2]}{}' \omega_m^{ij}} \tag{4}$$

where $\delta = 1, 2, \ldots, [q/2]$ and ω_m^{ij} is the Boltzman weight defined as

$$\omega_m^{ij} = \exp\left(\sum_{p=1}^{[q/2]}{}' K_p^{ij}\left(\cos\left(\frac{2\pi}{q} mp\right) - 1\right)\right) \tag{5}$$

where $\sum'$ indicates that the last term is not multiplied by 2 for the even values of q.

This vector transmissivity generalizes the scalar quantity that is used for the Ising (reproduced for $q = 2$) for q-state Potts (recovered as $t_1^{ij} = t_2^{ij} = \cdots = t_{q-1}^{ij}$) [24] and for the A–T (which corresponds to $q = 4$) [9]. Since the dual variables of t_δ are ω_δ, only the

$$1 + 2\sum_{m=1}^{[q/2]}{}' t_m^{ij} \cos\left(\frac{2\pi}{q} m\delta\right) \geqslant 0$$

region is physically meaningful, corresponding to the real values of K_δ.

The quenched bond-dilution is introduced by associating the probability law with the random variables $\{K_m^{ij}\}$,

$$P_K(\{K_m^{ij}\}) = p\prod_{m=1}^{[q/2]} \delta(K_m^{ij} - K_m) + (1-p)\prod_{m=1}^{[q/2]} \delta(K_m^{ij}) \tag{6}$$

where $0 \leqslant p \leqslant 1$ and $(t_1, t_2, \ldots, t_{[q/2]})$ are constants restricted to the ferromagnetic region. The probability distribution of the corresponding transmissivity-vector is given by

$$P_t(\{t_m^{ij}\}) = p\prod_{m=1}^{[q/2]} \delta(t_m^{ij} - t_m) + (1-p)\prod_{m=1}^{[q/2]} \delta(t_m^{ij})\,. \tag{7}$$

To generalize the BCM to the $Z(q)$ model, we will consider, at first, the pure case $p = 1$. The transmissivity $t^{(s)}(t^{(p)})$, corresponding to a series (parallel) array of two bonds, respectively, associated with $t^{(1)}$ and $t^{(2)}$, is given by

$$t_\delta^{(S)} = t_\delta^{(1)} t_\delta^{(2)} \qquad \delta = 1, 2, \ldots, [q/2] \qquad \text{(bonds in series)} \tag{8}$$

and

$$(t_\delta^{(p)}) = (t_\delta^{(1)})^D (t_\delta^{(2)})^D \qquad \delta = 1, 2, \ldots, [q/2] \qquad \text{(bonds in parallel)} \tag{9}$$

where the components of the dual transmissivity $\boldsymbol{t}^D$ are given by

$$t_\delta^D = \frac{1 + 2\sum_{m=1}^{[q/2]}{}' t_m \cos\left(\frac{2\pi}{q} m\delta\right)}{1 + 2\sum_{m=1}^{[q/2]}{}' t_m}\,. \tag{10}$$

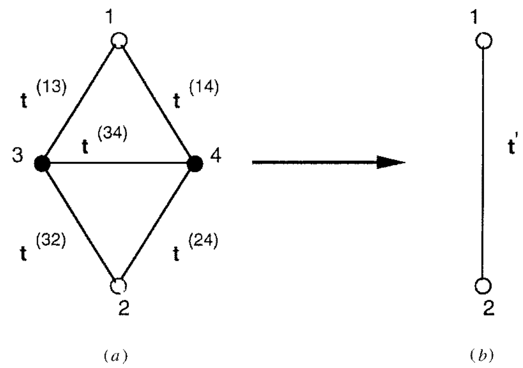

Figure 1. Self-dual two-terminal clusters used in the renormalization process. ○ (●) denotes the terminal (internal) sites.

To treat the Hamiltonian (2) we use the cluster RG transformation indicated in figure 1 (the convenient choice for square lattice) and renormalize the cluster indicated in figure 1(a) into a single bond indicated in figure 1(b). The RG recursive relations are constructed to preserve the two-body correlation functions, i.e. $\exp(-\beta H'_{1,2}) = \mathop{\mathrm{Tr}}_{3,4}(-\beta H_{1234})$, where $H'_{1,2}$ and H_{1234} are the Hamiltonians, respectively, associated with figures 1(a) and (b). ($H'_{1,2}$ includes an additive constant.) It is tedious to perform the trace, mainly for the large values of q.

The BCM makes it possible to calculate the equivalent transmissivity corresponding to any two-terminal array reductible whether or not in series/parallel sequences.

To obtain the RG equations we generalize the BCM to the $Z(q)$ model. The equivalent transmissivity $\boldsymbol{t}'$ associated with an arbitrary two-terminal graph of the $Z(q)$ bonds has $[q/2]$ components which are determined by $t'_\delta = N_\delta(\{t^{(i)}\})/D(\{t^{(i)}\})$, $\delta = 1, 2, \ldots, [q/2]$. $\{t^{(i)}\}$ denotes the set of transmissivities, respectively, associated with the bonds of the graph, and $N_\delta(\{t^{(i)}\})$ and $D(\{t^{(i)}\})$ are multilinear polynomials of the form $\sum_{n=0}^{[q/2]} A_n t_n^{(j)} = 0$ for an arbitrary jth bond. The quantities A_n depend on the set of transmissivities (denoted by $\{t^{(i)}\}'$) of the remaining bonds. Their determination depends on the symmetry of the model.

For some values of q the model is symmetric under some permutations. Then, if we denote by S the degree of such a symmetry, the total number of the pure phases is given by $N_\Phi = [q/2] + 1 - S$. For example, for $q = 5$, the model is symmetric under permutation $t_1 \to t_2$ then $S = 1$ which gives $N_\Phi = 2$ (ferromagnetic and paramagnetic phases) while for $q = 6$, $S = 0$ (no symmetry under permutation) the model has four phases, $N_\Phi = 4$, paramagnetic, ferromagnetic and two partially ordered phases. However, if q is a prime number, the model is symmetric under permutation $K_1 \to K_2 \to \cdots \to K_{[q/2]}$ which gives $S = [q/2] - 1$. Then, the performance of only two different operations on the jth bond, namely 'the break' ($t_1^{(j)} = t_2^{(j)} = \cdots = t_{[q/2]}^{(j)} = 0$) and the 'collapse' ($t_1^{(j)} = t_2^{(j)} = \cdots = t_{[q/2]}^{(j)} = 1$) determine completely the quantities A_n, using the symmetry of the model. It immediately follows for a prime number q:

$$
\begin{aligned}
N_\delta(\{t^{(i)}\}) &= \left(1 - \frac{1}{[q/2]} \sum_{m=1}^{[q/2]} t_m^{(j)}\right) N_\delta^{\mathrm{b}}(\{t^{(i)}\}') + \frac{1}{[q/2]} \sum_{m=1}^{[q/2]} t_m^{(j)} N_\delta^{\mathrm{c}}(\{t^{(i)}\}') \\
D(\{t^{(i)}\}) &= \left(1 - \frac{1}{[q/2]} \sum_{m=1}^{[q/2]} t_m^{(j)}\right) D^{\mathrm{b}}(\{t^{(i)}\}') + \frac{1}{[q/2]} \sum_{m=1}^{[q/2]} t_m^{(j)} D^{\mathrm{c}}(\{t^{(i)}\}')
\end{aligned}
\qquad (11)
$$

where N_δ^{b}, N_δ^{c}, D_δ^{b}, D_δ^{c} are the numerators and denominators of the 'broken' (b) and the 'collapsed' (c) graphs. In general, for any value of q, the $Z(q)$ model has $N_{\mathrm{pc}} = N_\Phi - 2$

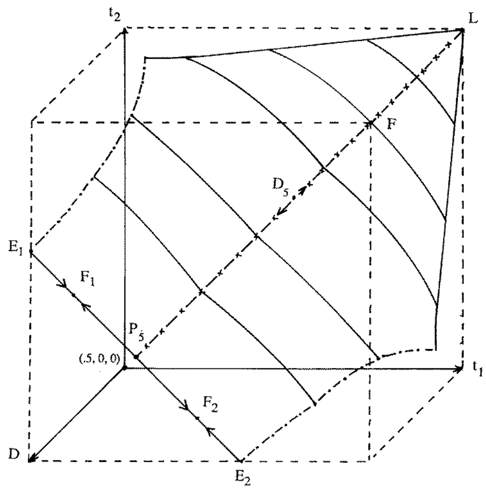

Figure 2. Phase diagram of the diluted $Z(5)$ model in the (p, t_1, t_2) space $p \geqslant p_c = \frac{1}{2}$. The model has two phase sinks: $F(1, 1, 1)$ and paramagnetic $D(1, 0, 0)$. The bifurcation fixed points $F_1(1, 0.127, 0.494)$, $F_2(1, 0.494, 0.127)$ and the five-state Potts fixed point $P_5\left(1, \frac{\sqrt{5}-1}{4}, \frac{\sqrt{5}-1}{4}\right)$ are located in the line invariant under duality E_1E_2 with $E_1\left(1, 0, \frac{\sqrt{5}-1}{2}\right)$ and $E_2\left(1, \frac{\sqrt{5}-1}{2}, 0\right)$. The diluted Potts fixed point is $D_5(0.6735, 0.6068, 0.6068)$ located on the diluted Potts line P_5L. The fixed point $L(0.5, 1, 1)$ is unstable. The physical region is bounded by the chain curve.

partially ordered phases. Then, the determination of the quantities A_n needs, in addition, the performance of the N_{pc} 'precollapsed' (the t_δ variables are zero for some values of δ while the remaining variables are $t_\delta = 1$) graphs. Then,

$$
\begin{aligned}
N_\delta(\{t^{(i)}\}) &= A^{b}(\{t^{(j)}\})N_\delta^{b}(\{t^{(i)}\}') + A^{c}(\{t^{(j)}\})N_\delta^{c}(\{t^{(i)}\}') + \sum_{k=1}^{N_{pc}} A_k^{bc}(\{t^{(j)}\})N_{\delta,k}^{bc}(\{t^{(i)}\}') \\
D(\{t^{(i)}\}) &= A^{b}(\{t^{(j)}\})D^{b}(\{t^{(i)}\}') + A^{c}(\{t^{(j)}\})D^{c}(\{t^{(i)}\}') + \sum_{k=1}^{N_{pc}} A_k^{bc}(\{t^{(j)}\})D_k^{bc}(\{t^{(i)}\}')
\end{aligned}
\tag{12}
$$

where $N_{\delta,k}^{bc}$ and D_k^{bc} are the numerators and the denominators of the 'precollapsed' (bc) graphs (the summation is over all the 'precollapsed' graphs which occur in the model) and A^b, A^c, A_k^{bc} are multilinear polynomials which depend on the jth bond. Their determination depends on the value of q. The knowledge of N_δ^b, N_δ^c, D^b, D^c, $N_{\delta,k}^{bc}$ and D_k^{bc} enables the calculation of N_δ and D. The transmissivities $\boldsymbol{t}^b$ and $\boldsymbol{t}^c$ are calculated by using the series and the parallel algorithms expressed in (8) and (9). Then, we obtain (see figure 2),

$$
t_\delta^{b} = \frac{1 + 2\sum_{n=1}^{[q/2]}{}' \cos\left(\frac{2\pi}{q}\delta n\right)(t_n^{b})^{D}}{1 + 2\sum_{n=1}^{[q/2]}{}' (t_n^{b})^{D}}
\tag{13}
$$

with

$$
(t_\delta^{b})^{D} = \frac{1+2\sum_{n=1}^{[q/2]}{}' \cos\left(\frac{2\pi}{q}\delta n\right)(t_n^{13}t_n^{32}+t_n^{14}t_n^{24}) + \left(2\sum_{n=1}^{[q/2]}{}' \cos\left(\frac{2\pi}{q}\delta n\right)(t_n^{13}t_n^{32})\right)\left(2\sum_{n=1}^{[q/2]}{}' \cos\left(\frac{2\pi}{q}\delta n\right)(t_n^{14}t_n^{24})\right)}{1+2\sum_{n=1}^{[q/2]}{}' (t_n^{13}t_n^{32}+t_n^{14}t_n^{24}) + \left(2\sum_{n=1}^{[q/2]}{}' (t_n^{13}t_n^{32})\right)\left(2\sum_{n=1}^{[q/2]}{}' (t_n^{14}t_n^{24})\right)}
$$

and

$$
t_\delta^{c} = \left[\frac{1 + 2\sum_{n=1}^{[q/2]}{}' \cos\left(\frac{2\pi}{q}\delta n\right)(t_n^{13,p})^{D}}{1 + 2\sum_{n=1}^{[q/2]}{}' (t_n^{13,p})^{D}}\right]\left[\frac{1 + 2\sum_{n=1}^{[q/2]}{}' \cos\left(\frac{2\pi}{q}\delta n\right)(t_n^{32})^{D}}{1 + 2\sum_{n=1}^{[q/2]}{}' (t_n^{13})^{D}}\right]
\tag{14}
$$

with

$$(t_\delta^{ij,p})^D = \frac{1+2\sum_{n=1}^{[q/2]}{}' \cos\left(\frac{2\pi}{q}\delta n\right)(t_n^{ij}+t_n^{ik})+\left(2\sum_{n=1}^{[q/2]}{}' \cos\left(\frac{2\pi}{q}\delta n\right)(t_n^{ij})\right)\left(2\sum_{n=1}^{[q/2]}{}' \cos\left(\frac{2\pi}{q}\delta n\right)(t_n^{ik})\right)}{1+2\sum_{n=1}^{[q/2]}{}' (t_n^{ij}+t_n^{ik})+\left(2\sum_{n=1}^{[q/2]}{}' (t_n^{ij})\right)\left(2\sum_{n=1}^{[q/2]}{}' (t_n^{ik})\right)}$$

$$k = \begin{cases} 4 & \text{for} \quad (i,j)=(1,3) \\ 1 & \text{for} \quad (i,j)=(3,2)\,. \end{cases}$$

The precollapsed terms are more complex, and have to be further reduced through the BCM (recursive use of the algorithm (12)). All the reducible graphs in series and parallel operations are straightforwardly calculated. Only the graphs which contain ($\{t_\delta = 0\}$ and $\{t_{\nu\neq\delta} = 1\}$) bonds resist until the very last step. The transmissivity of a graph which contains only bonds of the same nature, i.e. all the bonds have the same sets $\{t_\delta = 0\}$ and $\{t_{\nu\neq\delta} = 1\}$, itself satisfies $\{t_\delta = 0\}$ and $\{t_{\nu\neq\delta} = 1\}$ while the transmissivity of the graphs, containing mixed bonds, must be evaluated by calculating the partial trace over the internal spin (3 and 4) configurations. Such a tracing is simple since most variables t_δ in those graphs vanish and the remaining variables are $t_\alpha = 1$.

The renormalized parameters obey a new probability distribution which has the form

$$\begin{aligned} P(\{t_\delta^{ij}\}) = {} & p^5 \prod_{n=1}^{[q/2]} \delta(t_n^{ij} - t_n^{(a_1)}) + p^4(1-p)\left[\prod_{n=1}^{[q/2]} \delta(t_n^{ij} - t_n^{(a_2)}) + 4\prod_{n=1}^{[q/2]} \delta(t_n^{ij} - t_n^{(a_3)})\right] \\ & +2P^3(1-p)^2 \prod_{n=1}^{[q/2]} \delta(t_n^{ij} - t_n^{(a_4)}) + [6p^3(1-p)^2 + 2p^2(1-p)^3] \\ & \times \prod_{n=1}^{[q/2]} \delta(t_n^{ij} - t_n^{(a^5)}) + [2p^3(1-p)^2 + 8p^2(1-p)^3 + 5p(1-p)^4 + (1-p)^5] \\ & \times \prod_{n=1}^{[q/2]} \delta(t_n^{ij}) \end{aligned} \qquad (15)$$

where ($\{t_n^{(a_1)}\}$, $\{t_n^{(a_2)}\}$, $\{t_n^{(a_3)}\}$, $\{t_n^{(a_4)}\}$, $\{t_n^{(a_5)}\}$) are functions of $\{t_\delta\}$ such that $t_\delta^{a_5} = t_\delta^2$ and $t_\delta^{(\alpha)} = N_\delta^{(\alpha)}/D^{(\alpha)}$ ($\alpha = a_1, a_2, a_3, a_4$) where $N_\delta^{(\alpha)}$ and $D^{(\alpha)}$ are given by (12). The numerators and the denominators of the 'broken', 'collapsed' and 'precollapsed' graphs, $\{N_\delta^{\rm b}\}$, $\{N_\delta^{\rm c}\}$, $D^{\rm b}$, $D^{\rm c}$, $\{N_{\delta,k}^{\rm bc}\}$; $\{D_k^{\rm bc}\}$ are given in the appendix. To determine the renormalized variables we shall approximate $P(\{t_\delta^{ij}\})$ to the binary law:

$$P'(\{t_m^{ij}\}) = p' \prod_{n=1}^{[q/2]} \delta(t_n^{ij} - t_n') + (1-p') \prod_{n=1}^{[q/2]} \delta(t_n^{ij})\,. \qquad (16)$$

In order to obtain the renormalized variables p' and $\{t_\delta'\}$ as functions of p and $\{t_\delta\}$ we equalize the lowest-order moments of $\{t_\delta^{(ij)}\}$ calculated with both distributions (15) and (16):

$$\begin{aligned} \langle t_\delta^{(ij)}\rangle p &= \langle t_\delta^{(ij)}\rangle p' \\ \langle t_\delta^{(ij)} t_\alpha^{(ij)}\rangle p &= \langle t_\delta^{(ij)} t_\alpha^{(ij)}\rangle p' \qquad (\alpha \neq \delta)\,. \end{aligned} \qquad (17)$$

The above equations yield the explicit RG recursive relations, i.e.

$$p', t_1', t_2', \ldots, t_{[q/2]}' = F_n(p, t_1, t_2, \ldots, t_{[q/2]}) \qquad (n = p', t_1', t_2', \ldots, t_{[q/2]}')\,. \qquad (18)$$

The iteration of (18) generates flux lines in the parameter space $(p, t_1, t_2, \ldots, t_{[q/2]})$ from which one can find the critical frontiers. The fixed points are obtained as a solution of

$$p = F_x(0, \{t_\delta\}) \qquad t_\delta = F_\delta(p, \{t_\delta\}) \qquad \delta = 1, 2, \ldots, [q/2]\,. \qquad (19)$$

The critical fixed points provide information about the criticality of the system. The correlation-length exponents ν_i are calculated from the relevant eigenvalues ($\lambda_i > 1$) of the Jacobian $\partial(p', t_1', t_2', \ldots, t_{[q/2]}')/\partial(p, t_1, t_2, \ldots, t_{[q/2]})$ at the critical fixed point. Then, we have $\nu_i = \ln b / \ln \lambda_i$ where b is the linear scaling factor (in our case $b = 2$). Whenever one has more than one relevant eigenvalue, it is useful to define the crossover exponent $\Phi_{ij} = \nu_i/\nu_j$ which measures the possible different critical behaviour that may appear as one moves away from a given critical point.

3. Results

$q = 5$

The ferromagnetic $Z(5)$ model has been investigated by several authors. The phase structure proposed by Wu [12] and Alcaraz and Köberle [5] is based on the duality transformation and symmetry considerations. They suggested that the phase diagram presents three phases: ferromagnetic phase (F), disordered phase (D) and a sopht phase. They argued that the last region is a spin–wave phase. Rujan *et al* [6], Nishimori [15] and Roomany and Wyld [16] obtained only the ferromagnetic phase and the disordered one. By exploring the finite-size scaling ideas and the conformal invariance of the critical infinite system, Alcaraz [13] showed that the massless spin–wave phase originates, for the $Z(5)$ model, at a bifurcation point. Bonnier *et al* [14] have obtained the same result.

Since the model is symmetric under permutation $t_1 \to t_2$ as q is a prime number only, 'broken' and 'collapsed' graphs occur. Then $t_\delta' = N_\delta/D$ with

$$\begin{aligned} N_\delta(\{t^{(i)}\}) &= [1 - \tfrac{1}{2}(t_1^{34} + t_2^{34})]N_\delta^{\mathrm{b}} + \tfrac{1}{2}(t_1^{34} + t_2^{34})]N_\delta^{\mathrm{c}} \\ D(\{t^{(i)}\}) &= [1 - \tfrac{1}{2}(t_1^{34} + t_2^{34})]D^{\mathrm{b}} + \tfrac{1}{2}(t_1^{34} + t_2^{34})]D^{\mathrm{c}} \end{aligned} \tag{20}$$

where N_δ^{b}, N_δ^{c}, D^{b} and D^{c} are given in the appendix by replacing $q = 5$.

The recursive relation given by (17) provides the phase diagram and flux lines shown in figure 2.

As expected, the $Z(5)$ model exhibits two different 'pure' phases F and D which are respectively characterized by an attractor in the (p, t_1, t_2) space, namely $(1, 1, 1)$ and $(1, 0, 0)$. The bifurcation points F_1 and F_2, in the subspace $p = 1$ (pure case), are stable on the line invariant under duality. Their region of stability appears to extend to a short distance beyond this line. This behaviour is reflected by plotting the function $g(t_1, t_2)$ (figure 3) defined by

$$g(t_1, t_2) = \begin{cases} \left[\displaystyle\sum_{\delta=1}^{2}(t_\delta' - t_\delta)^2\right]^{1/2} & \text{for} \quad (t_1, t_2) > (t_1^*, t_2^*) \\ -\left[\displaystyle\sum_{\delta=1}^{2}(t_\delta' - t_\delta)^2\right]^{1/2} & \text{for} \quad (t_1, t_2) < (t_1^*, t_2^*) \end{cases} \tag{21}$$

on the clock line. (t_1^*, t_2^*) is the clock fixed point F_1.

This behaviour indicates the emergence of a well simulated massless spin–wave-like phase which appears beyond a bifurcation point. However, this method produces the well known results [13, 14], namely that this phase occurs beyond a bifurcation point while the MKRG indicates the presence of such a phase only for $q \geqslant 6$. Within this method we reproduce some exact results for $p = 1$:

(i) The critical line $\mathrm{E}_1\mathrm{E}_2$ is a line invariant under duality.
(ii) The fixed point of the five-state Potts model, $P_5\left(\frac{\sqrt{5}-1}{4}, \frac{\sqrt{5}-1}{4}\right)$.

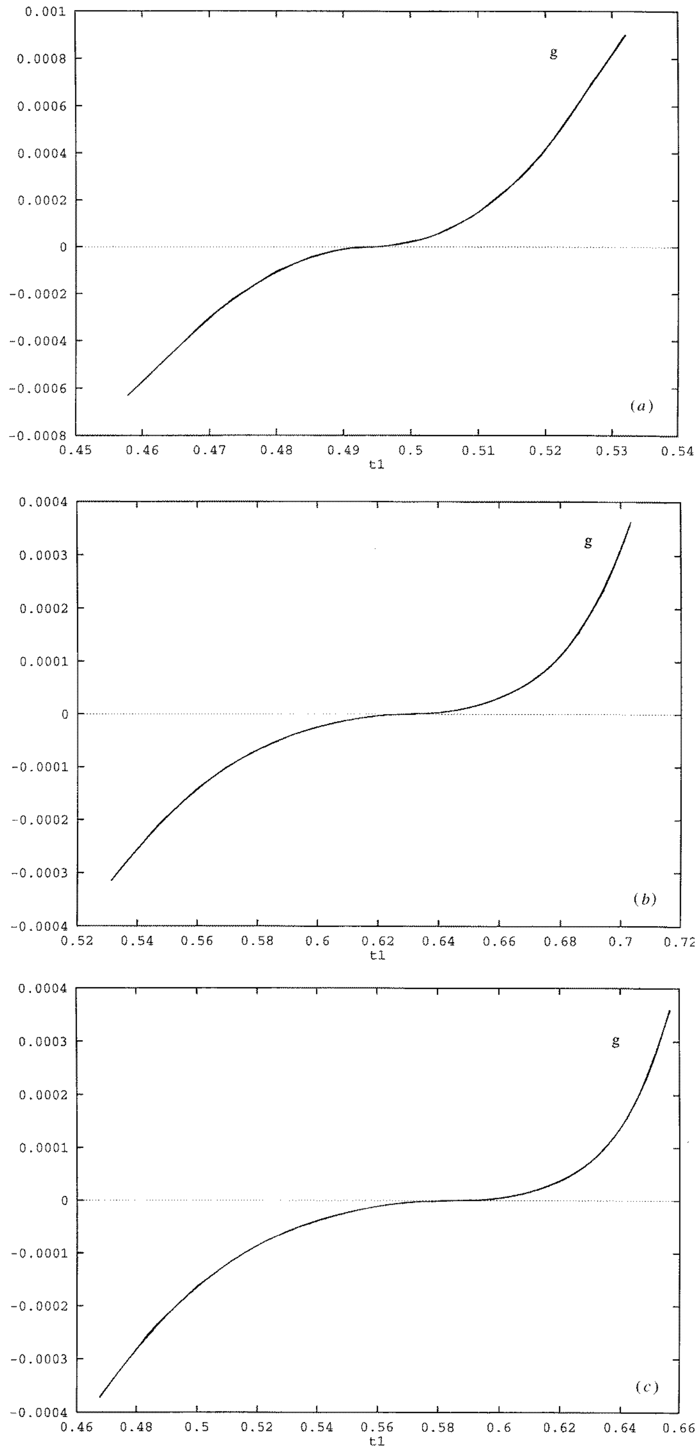

Figure 3. The g function versus t_1 on the line generated by the flow trajectory starting on the clock line: (a) $Z(5)$ model, (b) $Z(6)$ model and (c) $Z(7)$ model. A clear tendency to a line of fixed points, which is defined by $g = 0$, is observed.

The critical percolation concentration is $p_c = \frac{1}{2}$ for all values of K_2/K_1, confirming the conjecture proposed by Alcaraz and Tsallis [7]. The critical line $p = p_c$ presents one fixed point L (unstable), located at $(t_1, t_2) = (1, 1)$.

The invariant subspace $t_1 = t_2$ corresponds to the diluted five-state Potts model. It has a critical point, D_5, which is stable on the diluted Potts line P_5L. The q-state Potts model exhibits a first order transition in $d = 2$ dimension for $q > 4$ (exact result). But like most RG approaches, the fixed Potts point, P_5, undergoes a second-order phase transition. However, within this method the diluted fixed point, D_5, describes a crossover component since the pure fixed point, P_5, becomes unstable. It is known exactly that the five-state Potts model exhibits a first-order phase transition while the critical percolation fixed point L undergoes a second-order phase transition [17]. Then, we believe that an exact study of the diluted q-state Potts model will indicate that D_5 is tricritical point which is transformed into a crossover point within a real RG scheme.

The slope $a = \frac{1}{T_c}\frac{DT_c}{dp}\Big|_{p=1}$ obtained by using this method ($a = 1.233$) is in agreement with the exact result ($a_{ex} = 1.295$) [17].

$q = 6$

The phase structure of the pure two-dimensional $Z(6)$ model was discussed, in detail, by different authors [6, 30]. It can be parametrized, in terms of the $(3, 2)$ model of Domany and Riedel [30], by the identification

$$K_1 \cos\alpha(\Delta\sigma) + K_2 \cos 2\alpha(\Delta\sigma) + K_3 \cos 3\alpha(\Delta\sigma)$$
$$\to L_2 S_1 S_2 + L_3 \cos\frac{2\pi}{3}(\Delta m) + L_6 S_i S_j \cos\frac{2\pi}{3}(\Delta m) \tag{22}$$

where $\alpha = 2\pi/6$, $\Delta\sigma = \sigma_i - \sigma_j$, $\Delta m = m_i - m_j$, $\sigma_i = 0, 1, \ldots, 5$, $m_j = 0, 1, 2$, $S_i = \pm 1$.

Therefore, the model describes the coupled Ising and three-state Potts models. As described above, the renormalized variables t'_δ are given by $t'_\delta = N_\delta/D$ $(\delta = 1, 2, 3)$. To determine the quantities N_δ and D, we shall operate on the central bond of figure 1(*a*) and obtain the 'broken' $(t_1 = t_2 = t_3 = 0)$, the 'collapsed' $(t_1 = t_2 = t_3 = 1)$ and the 'precollapsed' $(t_{1,1} = 0, t_{2,1} = 1, t_{3,1} = 0)$, $(t_{1,2} = 0, t_{2,2} = 0, t_{3,2} = 1)$ graphs, respectively, indicated in figure 4. If we note $t^b = (t_1^b, t_2^b, t_3^b) \equiv (N_1^b/D^b, N_2^b/D^b, N_3^b/D^b)$, $t^c = (t_1^c, t_2^c, t_3^c) \equiv (N_1^c/D^c, N_2^c/D^c, N_3^c/D^c)$, $t_k^{bc} = (t_{1,k}^{bc}, t_{2,k}^{bc}, t_{3,k}^{bc}), \equiv (N_{1,k}^{bc}/D_k^{bc}, N_{2,k}^{bc}/D_k^{bc}, N_{3,k}^{bc}/D_k^{bc})$ $(k = 1, 2)$, the transmissivities, respectively, associated with the graphs of figure 4(*c*)–(*d*), the quantities N_δ, D that we are

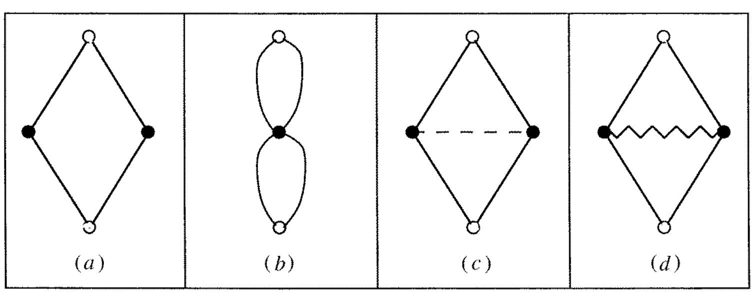

Figure 4. (*a*) broken, (*b*) collapsed, (*c*) precollapsed 1 and (*d*) precollapsed 2 graphs for the $Z(6)$ model obtained from that of figure 1(*a*), respectively, considering $t_1 = t_2 = t_3 = 0$; $t_1 = t_2 = t_3 = 1$; $t_1 = 0, t_2 = 0, t_3 = 1$ and $t_1 = 0, t_2 = 1, t_3 = 0$.

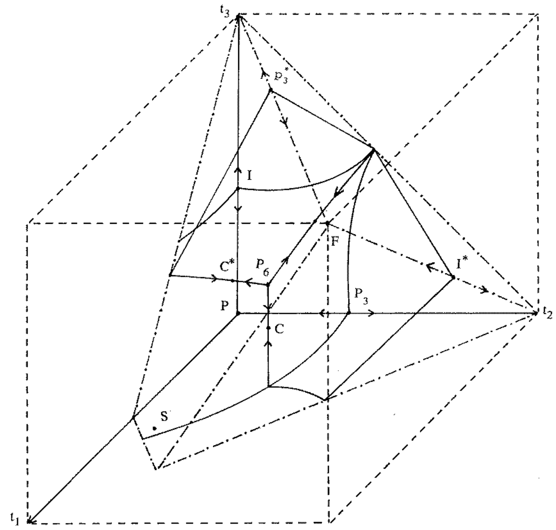

Figure 5. Phase diagram of the $Z(6)$ model in the (t_1, t_2, t_3) space for $p = 1$. The model has four phase sinks: ferromagnetic F(1, 1, 1), paramagnetic P(0, 0, 0), partially ordered phase $F_2(0, 1, 0)$ and partially ordered phase $F_3(0, 0, 1)$. The critical surfaces are described by the fixed points: I $(0, 0, \sqrt{2}-1)$ and $I^*(\sqrt{2}-1, 1, \sqrt{2}-1)$ (Ising fixed points); $P_3(0, \frac{\sqrt{3}-1}{2}, 0)$ and $P_3^*(\frac{\sqrt{3}-1}{2}, \frac{\sqrt{3}-1}{2}, 1)$ (three-state Potts fixed points) and S (clock fixed point). They meet on the critical line described by the fixed point C(0.318, 0.318, 0.173) and C^*(0.338, 0.219, 0.338) (cubic fixed points) and D$(0.1516, \frac{\sqrt{3}-1}{2}, \sqrt{2}-1)$ (decoupling fixed point). The six-state Potts fixed point $P_6(\frac{\sqrt{6}-1}{5}, \frac{\sqrt{6}-1}{5}, \frac{\sqrt{6}-1}{5})$ is localized in the invariant subspace $t_1 = t_2 = t_3$. The chain lines bound the physical region.

looking for are given by

$$\begin{aligned} N_\delta &= (1-(t_3^{34}+t_2^{34}-t_1^{34}))N_\delta^{\rm b} + t_1^{34}N_\delta^{\rm c} + (t_2^{34}-t_1^{34})N_{\delta,1}^{\rm bc} + (t_3^{34}-t_1^{34})N_{\delta,2}^{bc} \\ D &= (1-(t_3^{34}+t_2^{34}-t_1^{34}))D^{\rm b} + t_1^{34}D^{\rm c} + (t_2^{34}-t_1^{34})D_1^{\rm bc} + (t_3^{34}-t_1^{34})D_2^{\rm bc}. \end{aligned} \tag{23}$$

The quantities $N_\delta^{\rm b}$, $N_\delta^{\rm c}$, $D^{\rm b}$ and $D^{\rm c}$ are easily calculated by using the algorithm expressed by equations (8) and (9) as the respective graphs are reducible in series and parallel operations. $N_{\delta,1}^{\rm bc}$, $N_{\delta,2}^{\rm bc}$, $D_1^{\rm bc}$ and $D_2^{\rm bc}$ are obtained by using the algorithm (23) and the fact that a graph exclusively made by precollapsed bonds of the same nature (all of type 1 or type 2) is itself precollapsed, whereas those with mixed bonds are obtained by calculating the partial trace which is very simple to do. They are written in their explicit form in the appendix.

The phase diagram of the $Z(6)$ model, in the pure case ($p = 1$), is shown in figure 5. It is obtained by using the recursive relations (17). The model has four phase sinks: ferromagnetic, F, (completely broken symmetry), partially ordered phase, F_2, ($Z(2)$ symmetry), partially ordered phase, F_3, ($Z(3)$ symmetry) and paramagnetic phase, P, ($Z(6)$ symmetry). Each of them is characterized by an attractor in the (p, t_1, t_2, t_3) space, which are, respectively, (1, 1, 1, 1), (1, 0, 1, 0), (1, 0, 0, 1) and (1, 0, 0, 0). In the pure case ($p = 1$), and using this RG scheme, we reproduce all the known exact results (figure 5):

(i) The fixed points of the three-state Potts model P_3 and its dual P_3^*, the Ising Model I and its dual I^* and the cubic model C and its dual C^*. The lines II^* and $P_3P_3^*$ meet on the decoupling fixed point of the two coupled models, D.

(ii) All critical lines meet on the six-state Potts fixed point, $P_6 = \left(\frac{\sqrt{6}-1}{5}, \frac{\sqrt{6}-1}{5}\right)$ which is located in the invariant subspace $t_1 = t_2 = t_3$.

(iii) The fixed point S which defines the universality class of the clock model is stable within the plan invariant under duality (DIP). Moreover, its region of stability appears to extend to a short distance beyond the DIP. This behaviour is reflected by plotting the function (see figure 3) defined by

$$g(t_1, t_2, t_3) = \begin{cases} \left[\sum_{i=1}^{3}(t_i' - t_i)^2\right]^{1/2} & \text{for} \quad (t_1, t_2, t_3) > (t_1^*, t_2^*, t_3^*) \\ -\left[\sum_{i=1}^{3}(t_i' - t_i)^2\right]^{1/2} & \text{for} \quad (t_1, t_2, t_3) < (t_1^*, t_2^*, t_3^*) \end{cases} \tag{24}$$

on the clock line. (t_1^*, t_2^*, t_3^*) is the clock fixed point S. It indicates an emergency of a well simulated massless spin–wave phase beyond the clock fixed point S.

The critical percolation concentration is $p_c = \frac{1}{2}$ for all values of $K_\delta/K_1(\delta = 2, 3)$. The critical line $p = p_c$ presents three fixed points: L_1 (semi-stable), L_2 (semi-stable), and L_3 (unstable), respectively, located at $(t_1, t_2, t_3) = (0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 1)$. The invariant subspaces $t_1 = t_2 = 0$ and $(t_1 = t_3,\ t_2 = 1)$ correspond to the diluted Ising model. It is, for $p > p_c$, in the same universality class of the pure model. The diluted three-state Potts model is located in the invariant subspace $t_1 = t_3 = 0$. It has a critical point in between which attracts all the other points except P_3 and L_2. This critical point, $D_3 = (1, 0, 1, 0)$, reproduces the result obtained by Yeomans and Stinchcombe [19]. The pure fixed point becomes unstable and the system exhibits a crossover to a new critical regime described by the diluted fixed point. The transition remains second order but exhibits new values of the critical exponents. This behaviour is in accord with the Harris criterion [20]. The critical exponent α, for the diluted q-state Potts fixed point, is positive for all $q \geqslant 4$ while it is still negative (as for $q = 2$) for $q = 3$ [19]. This shows that within this RG technique, for $b = 2$, the appearance of the diluted fixed point for three-state Potts model does not exactly correspond to the change in the sign of α. We believe that α becomes positive for large values of the scale factor b. The three-state Potts and Ising models are invariant subspaces of the $Z(6)$ model. However, we reproduce all critical exponent α and slope $a = \frac{1}{T_c}\frac{dT_c}{dp}\Big|_{p=1}$ which are given in [19]. The slope of the diluted six-state Potts model is, $a = 1.190$, in agreement with the exact result ($a_{ex} = 1.114$) [17]. Since this method does not give the correct nature of the transition of the six-state Potts model, we obtain a fixed point, D_6, which characterizes the crossover exponent within this RG technique (see figure 6). D_6 must be a tricritical point as in the diluted five-state Potts model. The structure of the phase diagram of the diluted model is qualitatively similar to that obtained in case of $p = 1$.

$q = 7$

The $Z(7)$ model is symmetric under permutation $t_1 \to t_2 \to t_3$. However, the renormalized variables $t_\delta'(\delta = 1, 2, 3)$ can be written as function of N_δ and D variables such that $t_\delta' = N_\delta/D$, with

$$\begin{aligned} N_\delta &= [(1 - \tfrac{1}{3}(t_1^{34} + t_2^{34} + t_3^{34})]N_\delta^b + \tfrac{1}{3}(t_1^{34} + t_2^{34} + t_3^{34})N_\delta^c \\ D &= [(1 - \tfrac{1}{3}(t_1^{34} + t_2^{34} + t_3^{34})]D^b + \tfrac{1}{3}(t_1^{34} + t_2^{34} + t_3^{34})D^c\,. \end{aligned} \tag{25}$$

The N_δ^b, N_δ^c, D^b and D^c variables are given in the appendix.

One expects only two pure phases, ferromagnetic phase (F) and disordered phase (D) as in the case $q = 5$. They are, respectively, characterized by an attractor in the (p, t_1, t_2, t_3) space, namely $(1, 1, 1, 1)$ and $(1, 0, 0, 0)$. In the pure case $(p = 1)$ (figure 7), the phase boundaries are surfaces described by S_1, S_2 and S_3 fixed points representing a region of

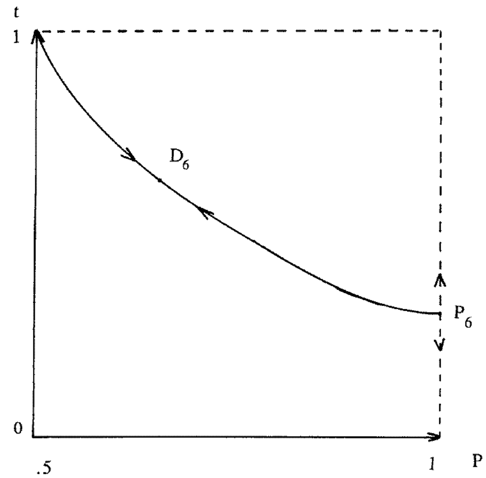

Figure 6. Phase diagram of the diluted six-state Potts model. $D_6(0.6539, 0.6310)$ is a diluted fixed point that is stable on the diluted line.

strong stability. This behaviour, which indicates the presence of a spin–wave-like phase, is reflected by plotting on the clock line the function $g(t_1, t_2, t_3)$ (figure 3) defined as follows:

$$g(t_1, t_2, t_3) = \begin{cases} \left[\sum_{i=1}^{3}(t_i' - t_i)^2\right]^{1/2} & \text{for } (t_1, t_2, t_3) > (t_1^*, t_2^*, t_3^*) \\ -\left[\sum_{i=1}^{3}(t_i' - t_i)^2\right]^{1/2} & \text{for } (t_1, t_2, t_3) < (t_1^*, t_2^*, t_3^*) \end{cases} \tag{26}$$

where (t_1^*, t_2^*, t_3^*) is the clock fixed point S_1.

The critical lines separating different critical surfaces are described by the unstable fixed points F_1, F_2 and F_3. The intersection of these lines is localized on the unstable fixed point P_7 which corresponds to the pure seven-state Potts model.

The critical percolation concentration is $p_c = \frac{1}{2}$ for all values of $K_\delta/K_1(\delta = 2, 3)$ as for all values of q. The critical line $p = p_c$ provides one fixed point L (unstable) which is located at $(t_1, t_2, t_3) = (1, 1, 1)$. The phase diagram for $p_c < p < 1$ is qualitatively similar to that obtained in the pure case. The diluted seven-state Potts model corresponds to the invariant subspace $t_1 = t_2 = t_3$. Within this RG technique, this model has a critical point D_7, which is stable on the diluted Potts line P_7L. It must be a tricritical point, but as this method does not give the correct nature of the transition, it describes a crossover component (figure 8).

This method gives, as for all values of q, a correct value of a slope $a = 1.240$ in comparison with the exact one $a_{ex} = 1.1965$ [17].

4. Conclusion

We have studied on the square lattice by means of a real-space renormalization-group technique the critical behaviour of the bond-diluted ferromagnetic $Z(q)$ model. The present method is exact on the hierarchical lattice generated by the Wheatstone bridge cluster (for $p = 1$). Then, we have generalized the break-collapse method, for $Z(q)$ models, which greatly simplifies the analytical task to establish the partial trace over the internal spin (3 and 4) configurations (figure 1). The procedure is operational and quite convenient as the tedious tracing algebraic calculations are automatically performed through elementary topological operations. The generalization of the BCM to $Z(q)$ models takes into account the

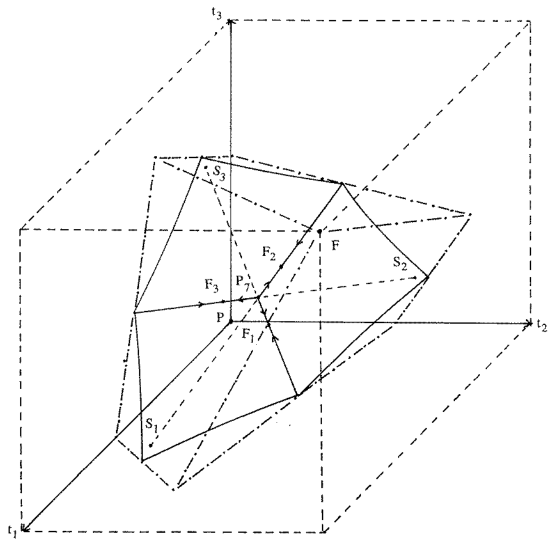

Figure 7. Phase diagram of the $Z(7)$ model in the (t_1, t_2, t_3) space for $p = 1$. F$(1, 1, 1)$ and P$(0, 0, 0)$ respectively describe the ferromagnetic and the paramagnetic phases. $S_1(0.6297, 0.1686, 0.0244)$, $S_2(0.1686, 0.0244, 0.6297)$ and $S_3(0.0244, 0.6297, 0.1686)$ describe the critical surfaces. $F_1(0.2967, 0.3295, 0.1958)$, $F_2(0.3295, 0.1958, 0.2967)$ and $F_3(0.1958, 0.2967, 0.3295)$ are bifurcation points. $P_7(\frac{\sqrt{7}-1}{6}, \frac{\sqrt{7}-1}{6}, \frac{\sqrt{7}-1}{6})$ is the seven-state Potts fixed point: the broken lines bound the physical region.

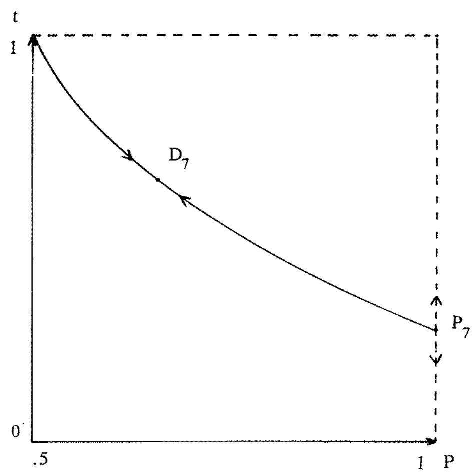

Figure 8. Phase diagram of the diluted seven-state Potts model. $D_7(0.6515, 0.6430)$ is a diluted fixed point that is stable on the dilute line.

symmetry of the latter. If S denotes the degree of the symmetry under permutation, we have $N_{pc} = [q/2] - S - 1$ partially ordered phases. However, the BCM gives in addition to the 'break' and 'collapsed' terms, N_{pc} 'precollapsed' ones. The first two terms are immediately calculated by using the series/parallel algorithm, while the last ones need to be reduced through the BCM until the very last step: the transmissivities of the graphs containing only bonds of the same nature reproduce themselves, while the graphs containing mixed bonds are evaluated by doing the partial trace. This latter is simple as most of the components of the transmissivity vector vanish and the remaining one is $t_\delta = 1$.

This method is able to produce, in the pure case $(p = 1)$, the critical frontiers of the most

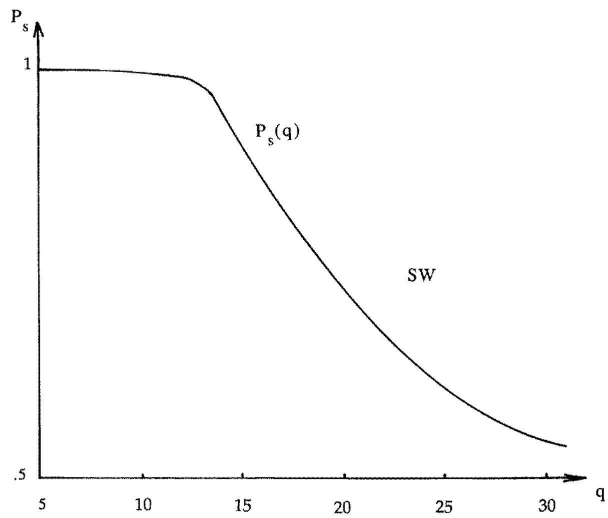

Figure 9. The threshold concentration $p_s(q)$ versus q above which the spin–wave (SW) occurs.

usual phases (e.g. paramagnetic, ferromagnetic, partially ordered) with very good precision. The phase diagrams are then in agreement with those obtained by several methods. Since the $Z(q)$ model reduces to the q-state Potts model in the invariant subspace $t_1 = t_2 = \cdots = t_{[q/2]}$ the method reproduces exactly all critical q-state Potts fixed points. It is known that for $d = 2$ the q-state Potts model exhibits a first-order phase transition for $q > 4$. The diluted model for $q \leqslant 4$ presents a crossover component, which is in agreement with the Harris criterion, described by a fixed point, D_q, that is stable on the critical line. If we take q as a continuous variable, the crossover starts from $q_{cr} = 2.672$.

It is known exactly that the percolation fixed point exhibits a second-order phase transition and the q-state Potts model, for $q > 4$, undergoes a first-order one. Then, we believe that the diluted q-state Potts fixed point D_q (for $q > 4$) will be a tricritical point. Meanwhile it appears, within this RG technique, as a crossover fixed point. However, for a given concentration p the square lattice becomes equivalent to a fractal system and the dimension of such a system, d_f, is less than the physical dimension $d = 2$. We know that $q_c(d)$ (the critical value of q such that for $q > q_c$ the model undergoes a first-order phase transition) decreases relative to dimension, d, of the system [31]. Consequently the critical line, for small values of p, is of second order and the diluted fixed point D_q is a tricritical. It will be located on the meeting of the second-order and the first-order critical line, respectively, described by the percolation fixed point $L(\frac{1}{2}, 1, 1)$ and the q-state Potts fixed point P_q.

The critical percolation concentration is, $p_c = \frac{1}{2}$ for all of $K_\delta/K_1(d = 2, 3, \ldots, [q/2])$, in agreement with the conjecture of Alcaraz and Tsallis [7]. It was shown that the pure $Z(q)$ model presents a massless spin–wave phase which evolves into Kosterlitz–Thouless (KT) phase [32] as $q \to \infty$, for $q \geqslant q_c$. Within this RG technique, we obtain $q_c = 5$ (exact result) while the Migdal–Kadanoff RG gives $q_c = 6$. To study the effect of the dilution on the spin–wave phase, we have plotted the function defined as follows:

$$g = \begin{cases} \left[(p' - p)^2 + \sum_{i=1}^{[q/2]} (t'_i - t_i)^2\right]^{1/2} & \text{for} \quad \boldsymbol{t} > \boldsymbol{t}^* \\ -\left[(p' - p)^2 + \sum_{i=1}^{[q/2]} (t'_i - t_i)^2\right]^{1/2} & \text{for} \quad \boldsymbol{t} < \boldsymbol{t}^* \end{cases} \tag{27}$$

on the clock line. $\boldsymbol{t}^*$ is the transmissivity vector of the clock fixed point.

We observe that a 'fixed' line occurs for $p > p_s(q)$ (concentration threshold), such that $p_c < p_s(q) < 1$. As $q \to \infty$, $p_s(q) \to p_c = \frac{1}{2}$ (figure 9). Consequently, the massless spin–wave phase resists even for the diluted $Z(q)$ model in a p concentration interval which becomes large for the large values of q.

In the limit $q \to \infty$ the $Z(q)$ model is reduced to the planar continum XY model. However, we can estimate the phase diagram of the diluted planar XY model. Indeed, on a line of the diluted clock model, the system presents a massless spin–wave phase for $p > p_s(q)$ and it remains in the same universality class of the pure one. Since the massless spin–wave phase evolves into a KT phase, the diluted planar XY model exhibits a phase transition between disordered and KT phases, for $p > p_c$, and it does not present a crossover component.

Acknowledgment

We would like to thank Professor A Benyoussef for many interesting discussions.

Appendix

The numerators and denominators of the 'broken' and the 'collapsed' transmissivities of the $Z(q)$ model are as follows:

$$N_\delta^{b,(a_1)} = \frac{1}{q}\left[(1+2\sum_{n=1}^{[q/2]}{}' t_n^2)^2 + 2\sum_{n=1}^{[q/2]}{}' \cos\left(\frac{2\pi}{q}n\delta\right)(1+2\sum_{l=1}^{[q/2]}{}' \cos\left(\frac{2\pi}{q}nl\right)t_1^2)^2\right]$$

$$D^{b,(a_1)} = \frac{1}{q}\left[(1+2\sum_{n=1}^{[q/2]}{}' t_n^2)^2 + 2\sum_{n=1}^{[q/2]}{}'(1+2\sum_{l=1}^{[q/2]}{}' \cos\left(\frac{2\pi}{q}nl\right)t_1^2)^2\right]$$

$$N_\delta^{c,(a_1)} = \frac{1}{q^2}\left[(1+2\sum_{n=1}^{[q/2]}{}' t_n)^2 + 2\sum_{n=1}^{[q/2]}{}' \cos\left(\frac{2\pi}{q}n\delta\right)(1+2\sum_{l=1}^{[q/2]}{}' \cos\left(\frac{2\pi}{q}nl\right)t_1)^2\right]^2$$

$$D^{c,(a_1)} = \frac{1}{q^2}\left[(1+2\sum_{n=1}^{[q/2]}{}' t_n)^2 + 2\sum_{n=1}^{[q/2]}{}'(1+2\sum_{l=1}^{[q/2]}{}' \cos\left(\frac{2\pi}{q}nl\right)t_1)^2\right]^2$$

$$t_\delta^{a_2} = \frac{N_\delta^{(a_2)}}{D^{(a_2)}} \qquad \text{with} \quad N_\delta^{(a_2)} = N_\delta^{(b,(a_1)} \qquad D^{(a_2)} = D^{b,(a_1)}$$

$$N_\delta^{b,(a_3)} = t_\delta^2 \qquad D^{b,(a_3)} = 1$$

$$N_\delta^{c,(a_3)} = \frac{t_\delta}{q^2}\left[(1+2\sum_{n=1}^{[q/2]}{}' t_n)^2 + 2\sum_{n=1}^{[q/2]}{}' \cos\left(\frac{2\pi}{q}n\delta\right)(1+2\sum_{l=1}^{[q/2]}{}' \cos\left(\frac{2\pi}{q}nl\right)t_1)^2\right]$$

$$D^{c,(a_3)} = \frac{1}{q^2}\left[(1+2\sum_{n=1}^{[q/2]}{}' t_n)^2 + 2\sum_{n=1}^{[q/2]}{}'(1+2\sum_{l=1}^{[q/2]}{}' \cos\left(\frac{2\pi}{q}nl\right)t_1)^2\right]$$

$$N_\delta^{b,(a_4)} = 0 \qquad D^{b,(a_4)} = 1 \qquad N_\delta^{c,(a_4)} = t_\delta^2 \qquad D^{c,(a_4)} = 1\,.$$

The numerators and denominators of the 'precollapsed' transmissivities of the $Z(6)$ model are as follows:

$$N_{1,1}^{bc,(a_1)} = 2(t_1 + t_1 t_2 + t_2 t_3)^2 \qquad N_{2,1}^{bc,(a_1)} = t_1^2(t_1 + 2t_3)^2 + t_2^2(2+t_2)^2$$

$$N_{3,1}^{bc,(a_1)} = 2(2t_1t_2 + t_3)^2$$
$$D_1^{bc,(a_1)} = (2t_1^2 + t_3^2)^2 + (1 + 2t_2)^2$$
$$N_{1,2}^{bc,(a_1)} = 2t_1^2(1 + 2t_2^2) + 2t_2t_3(t_1 + t_2t_3) \qquad N_{2,2}^{bc,(a_1)} = (t_1^2 + t_2^2)^2 + 2t_2^2 + 2t_1t_3(2t_2 + t_1t_3)$$
$$N_{3,2}^{bc,(a_1)} = 4(2t_1^2t_2^2 + t_3^2) \qquad D_2^{bc,(a_1)} = (1 + t_3^2)^2 + 2(t_1^2 + t_2^2)^2$$
$$N_{1,1}^{bc,(a_3)} = t_1(t_1 + t_1t_2 + t_2t_3) \qquad N_{2,1}^{bc,(a_3)} = t_2^2(2 + t_2) \qquad N_{3,1}^{bc,(a_3)} = t_3(2t_1t_2 + t_3)$$
$$D_1^{bc,(a_3)} = 1 + 2t_2^2$$
$$N_{1,2}^{bc,(a_3)} = t_1(t_1 + t_2t_3) \qquad N_{2,2}^{bc,(a_3)} = t_2(t_2 + t_1t_3) \qquad N_{3,2}^{bc,(a_3)} = 2t_3^2 \qquad D_2^{bc,(a_3)} = 1+t_3^2\,.$$

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